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LETTER TO THE EDITOR

On a subclass of Ince equations

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Abstract. We present a two-parameter family of Ince equations for which the problem of the periodicity of the general integral is reducible entirely to the solution of a quadratic equation.

The class of second-order, homogeneous, differential equations,

$$\frac{d^2y}{dt^2}+\frac{\alpha+\beta\cos 2t+\gamma\cos 4t}{(1+a\cos 2t)^2}y=0 \tag{1}$$

where $|a|<1$, is a four-parameter family of Hill's equations (with coefficients periodic of period π) called Ince equations by Magnus and Winkler [1] or generalised Ince equations by Arscott [2]. Their importance lies in the fact that the coexistence problem for such equations is solvable; that is, once it is known that there exists a particular integral of (1) with period $n\pi$, $n\in\mathbb{N}$, one can decide by purely algebraic means whether the general integral is periodic. For $n\geqslant 3$ this question can be resolved for the general Hill's equation by application of Floquet's theorem [1, 3]: the existence of an integral of period $n\pi$, $n\geqslant 3$, implies that the general solution is of this period. Hence interest focuses on the cases $n=1$ and $n=2$. Amongst the results proved in [1], the following is the simplest. By a transformation of the form $y\to(1+a\cos 2t)^{\eta}y$ we may rewrite (1) as,

$$(1+a\cos 2t)\frac{d^2y}{dt^2}+b\sin 2t\frac{dy}{dt}+(c+d\cos 2t)y=0 \tag{2}$$

and we define a quadratic function $Q(\mu)=2a\mu^2-b\mu-\frac{1}{2}d$. Then theorem 7.1 of [1] states that a necessary condition for the general integral of (1) or (2) to be periodic of period $\pi(2\pi)$ is that $Q(\mu)$ have a root in $\mathbb{Z}$ $(\mathbb{Z}+\frac{1}{2})$.

In this letter we consider a non-trivial subclass of equations of type (1) for which a much stronger result can be obtained. The method of proving this result involves embedding (1) in a coupled system of second-order, *nonlinear*, *autonomous* equations. This coupled system is a special case of a Ermakov system [4], possessing an invariant which can be written in such a way as to make the periodicity question for this subclass of Ince equation trivial.

The class of Ince equation with which we are concerned has the form:

$$\frac{d^2y}{dt^2}+\left(1+\frac{\alpha'}{(1+a\cos 2t)^2}\right)y=0 \tag{3}$$

and is a two-parameter (α' and a) family. The denominator of the π-periodic coefficient can be regarded as arising from the Pinney equation [5]:

$$\frac{d^2x}{dt^2}+x=\delta x^{-3} \tag{4}$$

the general integral of which is expressible in terms of the general integral of the linear version ($\delta=0$) and is written $x(t)=B(1+a\cos(2t+c))^{1/2}$ provided B and a satisfy the relation:

$$B^4(1-a^2)=\delta. \tag{5}$$

Since δ is at our disposal we choose it to be $1-\alpha' B^4$. Hence we may replace (3) by the coupled, autonomous pair of nonlinear equations:

$$\begin{aligned}\frac{d^2y}{dt^2}+y&=-\frac{\alpha' B^4}{x^4}y\\ \frac{d^2x}{dt^2}+x&=\frac{\delta}{x^3}\end{aligned} \tag{6}$$

where the constant c in $x(t)$ is clearly immaterial. Equation (6) is an example of an Ermakov system [4], more particularly of a coupled Pinney system [6] and therefore possesses two features that should be remarked upon. Firstly the system possesses a first integral (the so-called Ray–Reid invariant) which, in this case, is

$$I=\tfrac{1}{2}(x\dot{y}-y\dot{x})^2+\tfrac{1}{2}(y/x)^2. \tag{7}$$

Secondly, the system (6), along with a very large class of other Ermakov systems, can be reduced to a pair of linear equations [7]. The linearisation procedure is non-trivial but for systems of the form (6) it results in the simplest possible autonomous equations

$$\psi''+\delta\psi=0 \qquad x''+x=0. \tag{8}$$

Whilst it can be shown that commensurability of the periods of the integrals of this pair of independent linear equations (in this case $\delta^{1/2}\in\mathbb{Q}$) is equivalent to the periodicity of the general integral of the nonlinear, coupled Pinney system [8], it is simpler in the present instance to exploit the invariant (7) and bypass the linearisation (8). One may regard the present method as an application of the nonlinear superposition rule described by Ray and Reid [4], in a circumstance where it is explicit. We rewrite (7) in terms of dependent variables x and $z=y/x$ to give

$$\frac{dt}{x(t)^2}=\frac{dz}{\sqrt{2I-z^2}}. \tag{9}$$

From the expression for the invariant we see that z is bounded by $\pm\sqrt{2I}$. The obvious substitution $z=\sqrt{2I}\sin\theta$ gives

$$\frac{dt}{B^2(1+a\cos 2t)}=d\theta. \tag{10}$$

Since $|a|<1$ the integral of the LHS of (10) is augmented over one period of its integrand by an amount $\pi B^{-2}(1-a^2)^{-1/2}=\pi\delta^{-1/2}$. Hence over n periods of the coefficient in (3), $y(t+n\pi)=x(t+n\pi)z(t+n\pi)=\sqrt{2I}x(t)\sin(\theta(t)+n\pi\delta^{-1/2})$ and y is periodic of period $n\pi$ iff $n\delta^{-1/2}=2m$, i.e. $\delta^{1/2}=n/2m$. Now write (3) in the form (2): $b=-4\eta a$, $c=(2\eta-1)^2$ and $d=a(1-4\eta^2)$ provided η is chosen to satisfy $4\eta(\eta-1)(1-a^2)=\alpha'$.

Using (5) and the choice of δ this last condition is satisfied with $\eta = \frac{1}{2} - \frac{1}{2}\delta^{-1/2}$ and the quadratic function, $Q(\mu)$, introduced earlier is

$$Q(\mu) = 2a(\mu - \tfrac{1}{2}\delta^{-1/2})(\mu - \tfrac{1}{2}\delta^{-1/2} + 1). \tag{11}$$

Consequently we state the result.

Theorem. The restricted family of Ince equations (3) has general solution of period $n\pi$ iff the roots of the corresponding quadratic are both rational numbers differing by unity and of denominator n when expressed in lowest terms.

It should be noted that this theorem is consistent with the result quoted earlier. It should further be noted that the result extends unaltered to nonlinear Ince equations of the form

$$\frac{d^2y}{dt^2} + \left[1 + \frac{\alpha'}{(1 + a\cos 2t)^2}\right] y = \beta / y^3. \tag{12}$$

That this is so follows from the fact that (12) is, like (4), a Pinney equation and its general solution is expressible as

$$y(t) = (Ey_1^2 + 2Fy_1y_2 + Gy_2^2)^{1/2} \tag{13}$$

where $EG - F^2 = \beta$ and y_1, y_2 are linearly independent solutions of the equation (12) with $\beta = 0$ and having unit Wronskian.

Generally speaking in the study of Hill's equations one must deal, in discussing the existence of periodic solutions, with infinite determinants. It is not at all obvious *a priori* that equations of the type discussed in this letter should be any more tractable in this respect. The fact that they are so is because of an apparent coincidence allowing the time dependence of the coefficient to be removed at the expense of introducing a coupled nonlinear system but a system which is solvable by virtue of its possessing a first-order invariant which can be separated. Clearly one could write down systems tailored to the method; it is to be hoped, however, that it might prove applicable to further classes of equation of independent interest.

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